THE SIMPLE GROUP OF ORDER 168 AND K3 SURFACES

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Dedicated to Herr Professor Doctor Hans Grauert on the occasion of his seventieth birthday

Abstract. The aim of this note is to characterize a K3 surface of Klein-Mukai type in terms of its symmetry.

Introduction

The group $L_2(7)$ is by the definition the projectivized special linear group $PSL(2, \mathbf{F}_7)$ and is generated by the three projective transformations of $\mathbf{P}^1(\mathbf{F}_7)$ of order 7, 3, 2:

$$\alpha : x \mapsto x + 1; \beta : x \mapsto 2x; \gamma : x \mapsto -x^{-1},$$

where the coefficient 2 in β is a generator of the cyclic group $(\mathbf{F}_7^{\times})^2 (\simeq \mu_3)$. (See for instance [CS, Chapter 10].) As well-known, this group is of order 168 and is characterized as the second smallest non-commutative simple group.

One of interesting connections between $L_2(7)$ and complex algebraic geometry goes back to the result of the great German mathematicians Hurwitz and Klein in Göttingen: $|L_2(7)| = 84(3 - 1)$ is the largest possible order of a group acting on a genus-three curve and the so called Klein quartic curve

$$C_{168} = \{x_1x_2^3 + x_2x_3^3 + x_3x_1^3 = 0\} \subset \mathbf{P}^2$$

is the unique genus-three curve admitting an $L_2(7)$-action. The action of $L_2(7)$ on C_{168} is the projective transformation induced by (one of two essentially the same) 3-dimensional irreducible representation V_3 of $L_2(7)$ given by

$$\alpha \mapsto \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7^4 \end{pmatrix}; \quad \beta \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \gamma \mapsto \frac{-1}{\sqrt{-7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix},$$

where $\zeta_7 = \exp(2\pi\sqrt{-1}/7)$, $a = \zeta_7^2 - \zeta_7^5$, $b = \zeta_7 - \zeta_7^6$, $c = \zeta_7^4 - \zeta_7^3$, and the branch of $\sqrt{-7}$ is chosen so that $\sqrt{-7} = \zeta_7 + \zeta_7^2 + \zeta_7^4 - \zeta_7^3 - \zeta_7^5 - \zeta_7^6$. The other 3-dimensional irreducible

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representation is the composition of the representation V_3 with the outer automorphism of $L_2(7)$ given by $\alpha \mapsto \alpha^{-1}$, $\beta \mapsto \beta$ and $\gamma \mapsto \gamma$ (see [ATLAS] and [Bu, Section 267]). The Klein curve C_{168} together with $L_2(7)$-action also appears in the McKay correspondece problem [Ma] and a classification of Calabi-Yau threefolds [Og].

Our interest in this note is a relation between $L_2(7)$ and K3 surfaces.

Throughout this note, a *K3 surface* means a simply-connected smooth complex *algebraic* surface X with a nowhere vanishing holomorphic 2-from ω_X. We call an automorphism $g \in \mathrm{Aut}(X)$ *symplectic* if $g^*\omega_X = \omega_X$. According to Mukai's classification [Mu1], there are eleven maximum finite groups acting on K3 surfaces symplectically, and among them, there appear two simple groups: the group $L_2(7)$ and the alternating group A_6 of degree 6. In the same paper, Mukai also gives a beautiful example of K3 surface X_{168} with $L_2(7)$-action, where

$$X_{168} = \{x_0^4 + x_1x_2^3 + x_2x_3^3 + x_3x_1^3 = 0\} \subset \mathbf{P}^3.$$

This is the cyclic cover of $\mathbf{P}^2$ of degree 4 branched along the Klein quartic curve C_{168}. Here, the action of $L_2(7)$ on X_{168} is naturally induced by the action on C_{168}. Note that this X_{168} now admits a larger group action of $L_2(7) \times \mu_4$, where μ_4 is the Galois group of the covering. On the other hand, the smooth plane curve H_{168} of degree 6 defined by $\{5x_1^2x_2^2x_3^2 - x_1^5x_2 - x_2^5x_3 - x_3^5x_1 = 0\} \subset \mathbf{P}^2$ – the zero locus of the Hessian of the Klein quartic curve – is also invariant under the same $L_2(7)$-action on $\mathbf{P}^2$. So, the K3 surface

$$X'_{168} = \{y^2 = 5x_1^2x_2^2x_3^2 - x_1^5x_2 - x_2^5x_3 - x_3^5x_1\} \subset \mathbf{P}(1,1,1,3),$$

i.e. the double cover of $\mathbf{P}^2$ branched along H_{168}, also admits an $L_2(7)$-action. However, it will turn out that these two K3 surfaces X_{168} and X'_{168} are not isomorphic to each other (see **Remark (2.12)**). Therefore, K3 surfaces having $L_2(7)$-action are no more unique and it is of interest to characterize the Klein-curve-like K3 surface X_{168} in a flavour similar to that of Hurwitz and Klein. This is the aim of this short note.

Throughout this note, we set $G := L_2(7)$. Our main observation is as follows:

Main Theorem. *Let X be a K3 surface. Assume that $G \subset Aut(X)$. Let $\tilde{G}$ be a finite subgroup of $Aut(X)$ such that $G \subset \tilde{G}$. Then,*

(1) *$\tilde{G}/G$ is a cyclic group of order 1, 2, 3, or 4; and*
(2) *if $\tilde{G}/G$ is of the maximum order 4, then $(X, \tilde{G})$ is isomorphic to the Klein-Mukai pair $(X_{168}, L_2(7) \times \mu_4)$.*

Here an isomorphism means an equivariant isomorphism with respect to group actions.

The main difference between genus-three curves and K3 surfaces is that there are no canonical polarizations on K3 surfaces. In other words, we do not know *a priori* which K3 surfaces are quartic K3 surfaces or which polarizations are invariant under the group action. Indeed, the determination of the invariant polarization for $\tilde{G}$ – this will turn

out to be of degree four if $|\tilde{G}/G| = 4$ (**Claim (2.10)**) – is the most crucial part in this note. Besides Mukai's pioneering work, we are much inspired by a series of Kondo's work [Ko1,2] on a lattice theoretic proof of Mukai's classification and the determination of the K3 surface with the largest finite group action as well as the action. Especially we will fully exploit his brilliant idea of studying invariant lattices through an embedding of their orthogonal complements into some Niemeier lattices. This enables us to relate the problem with the Mathieu group M_{24} and the binary Golay code $\mathcal{C}_{24}$ (Section one) and provides a very powerful tool in calculating the discriminants of the invariant lattices $H^2(X, \mathbf{Z})^G$ also in our setting. Combining this with the additional group action $\tilde{G}/G$, we shall determine the invariant polarization in the maximum case $|\tilde{G}/G| = 4$. Once we find the invariant polarization in a lattice-theoretic way, we can continue the proof by coming back to more algebro-geometric arguments. One of the advantages of the algebro-geometric argument is perhaps that we can then express the K3 surface and the group action in a very concrete way as in the Theorem.

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1. The Niemeier Lattices

In this section, we recall some basic facts on the Niemeier lattices needed in our arguments. Our main reference concerning Niemeier lattices and their relations with Mathieu groups is [CS, Chapters 10, 11, 16, 18].

(1.1). In this note, the even *negative* definite unimodular lattices of rank 24 are called Niemeier lattices. (We changed the sign from positive into negative.) There are exactly 24 isomorphism classes of the Niemeier lattices and each isomorphism class is uniquely determined by its root lattice N_2, i.e. the sublattice generated by all the roots, the elements x with $x^2 = -2$. Except the so called Leech lattice which contains no roots, the other 23 lattices are the over-lattices of their root lattices, which are:

$$A_1^{\oplus 24},\ A_2^{\oplus 12},\ A_3^{\oplus 8},\ A_4^{\oplus 6},\ A_6^{\oplus 4},\ A_8^{\oplus 3},\ A_{12}^{\oplus 2},\ A_{24},$$
$$D_4^{\oplus 6},\ D_6^{\oplus 4},\ D_8^{\oplus 3},\ D_{12}^{\oplus 2},\ D_{24},\ E_6^{\oplus 4},\ E_8^{\oplus 3},\ A_5^{\oplus 4} \oplus D_4,\ A_7^{\oplus 2} \oplus D_5^{\oplus 2},$$
$$A_9^{\oplus 2} \oplus D_6,\ A_{15} \oplus D_9,\ E_8 \oplus D_{16},\ E_7^{\oplus 2} \oplus D_{10},\ E_7 \oplus A_{17},\ E_5 \oplus D_7 \oplus A_{11}.$$

We denote the Niemeier lattices N whose root lattices are $A_1^{\oplus 24}$, $A_2^{\oplus 12}$ and so on by $N(A_1^{\oplus 24})$, $N(A_2^{\oplus 12})$ and so on.

(1.2). In what follows, we regard the set of roots $R := \{r_i | 1 \le i \le 24\}$ corresponding to the vertices of the Dynkin diagram, as the set of the simple roots of N. Denote by $O(N)$ (resp. by $O(N_2)$) the group of isometries of N (resp. of N_2) and by $W(N)$ the Weyl group generated by the reflections given by the roots of N. Here $O(N) \subset O(N_2)$ and $W(N)$ is a normal subgroup of both $O(N)$ and $O(N_2)$. The invariant hyperplanes of the reflections divide $N \otimes \mathbf{R}$ into (finitely many) chambers. Then, each chamber is a fundamental domain of the action of $W(N)$ and the quotient group $S(N) := O(N)/W(N)$ is identified with a subgroup of symmetry of the distinguished chamber $\mathcal{C} := \{x \in N \otimes \mathbf{R} | (x, r) > 0 r \in R\}$ and also a subgroup of a larger group $S_{24} = \mathrm{Aut}_{\mathrm{set}}(R)$.

The groups $S(N)$ are very explicitly calculated in [CS, Chapters 18, 16]. (See also [Ko1].) The following is a part of the results there:

Lemma (1.3) [CS, Chapters 18, 16]. *Let N be a non-Leech Niemeier lattice. Then,*
1. $S(N) = M_{24}$ *if* $N = N(A_1^{\oplus 24})$;
2. $S(N) = C_2 \rtimes (C_2^{\oplus 3} \rtimes L_3(2))$ *if* $N = N(A_3^{\oplus 8})$; *and*
3. *for other N, the order $|S(N)|$ is not divisible by* 7. $\square$

Let us add a few remarks about the groups appearing in the Lemma above. The next (1.4) and (1.5) are concerned with the first case (1) and (1.6) is for the second case (2).

(1.4). Observe that $A_1^{\oplus 24} \subset N(A_1^{\oplus 24}) \subset (A_1^{\oplus 24})^*$ and that

$$(A_1^{\oplus 24})^*/A_1^{\oplus 24} = \oplus_{i=1}^{24} \mathbf{F}_2 \overline{r}_i \simeq \mathbf{F}_2^{\oplus 24}.$$

Here $\overline{r}_i := r_i/2 \,\mathrm{mod} \mathbf{Z} r_i$ is the standard basis of the i-th factor $(A_1)^*/A_1$. We also identify

$$S_{24} = \mathrm{Aut}_{\mathrm{set}}(R) = \mathrm{Aut}_{\mathrm{set}}(\{\overline{r}_i\}_{i=1}^{24}).$$

The linear subspace of $(A_1^{\oplus 24})^*/A_1^{\oplus 24}$

$$\mathcal{C}_{24} := N(A_1^{\oplus 24})/A_1^{\oplus 24} \simeq \mathbf{F}_2^{\oplus 12}$$

encodes the information which elements of $(A_1^{\oplus 24})^*$ lie in $N(A_1^{\oplus 24})$. Besides this role, this subspace $\mathcal{C}_{24}$ carries the structure of the binary self-dual code of Type II with minimal distance 8, called the (extended) binary Golay code.

Among many equivalent definitions, the Mathieu group M_{24} of degree 24 is defined to be the subgroup of S_{24} preserving $\mathcal{C}_{24}$, i.e.

$$M_{24} := \{\sigma \in S_{24} | \sigma(\mathcal{C}_{24}) = \mathcal{C}_{24}\}.$$

As well-known, M_{24} is a simple group of order $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 16 \cdot 3$ and acts on the set $\{\overline{r}_i\}_{i=1}^{24}$ as well as on R quintuply transitively.

Let $\mathcal{P}(R)$ be the power set of R, i.e. the set consisting of the subsets of R. Then, $\mathcal{P}(R)$ bijectively corresponds to the set $(A_1^{\oplus 24})^*/A_1^{\oplus 24}$ by:

$$\iota : \mathcal{P}(R) \ni A \mapsto \overline{r}_A := \frac{1}{2}\sum_{r_j \in A} r_j \bmod A_1^{\oplus 24} \in (A_1^{\oplus 24})^*/A_1^{\oplus 24}.$$

Set $\mathcal{E} := \iota^{-1}(\mathcal{C}_{24})$. Then $A \in \mathcal{E}$ if and only if $\frac{1}{2}\sum_{r_j \in A} r_j$ is in $N(A_1^{\oplus 24})$. Moreover, it is known that $\emptyset, R \in \mathcal{E}$ and that if $A \in \mathcal{E}$ ($A \neq R$, $\emptyset$) then $|A|$ is either 8, 12, or 16. We call $A \in \mathcal{E}$ an Octad (resp. a Dodecad) if $|A| = 8$ (resp. 12). Note that $B \in \mathcal{E}$ with $|B| = 16$ is then the complement of an Octad (in R), i.e. B is of the form $R - A$ for some Octad A. There are exactly 759 Octads.

The following fact called the Steiner property $St(5, 8, 24)$ and its proof are both needed in the proof of our main result:

Fact (1.5) (the Steiner property). *For each 5-elemet subset S of R, there exists exactly one Octad O such that $S \subset O$.*

Proof. Since M_{24} is quintuply transitive on R, there exists an Octad O such that $S \subset O$. Let O_1 and O_2 be two Octads. Then, by the definition, their symmetric difference $(O_1 - O_2) \cup (O_2 - O_1)$ is also an element of $\mathcal{E}$. Thus, $|(O_1 - O_2) \cup (O_2 - O_1)|$ is either 0, 8, 12, or 16 and we have that $|O_1 \cap O_2|$ is either 8, 4, 2 or 0. Therefore, if $S \subset O_1$ and $S \subset O_2$, then one has $|O_1 \cap O_2| \geq 5$, whence $|O_1 \cap O_2| = 8$. This means that $O_1 = O_2$. $\square$

(1.6). In the second case, we identify (non-canonically) the set of eight connected components of the Dynkin diagram $A_3^{\oplus 8}$ with the three-dimensional linear space $\mathbf{F}_2^{\oplus 3}$ over $\mathbf{F}_2$ by letting one connected component to be 0. The group $C_2 \rtimes (C_2^{\oplus 3} \rtimes L_3(2))$ is the semi-direct product, where C_2 interchanges the two edges of all the components, $C_2^{\oplus 3}$ is the group of the parallel transformations of the affine space $\mathbf{F}_2^{\oplus 3}$ and $L_3(2) (\simeq L_2(7))$ is the linear transformation group of $\mathbf{F}_2^{\oplus 3}$ which fixes (point wise) the three simple roots in the identity component.

2. Proof of the main Theorem

In what follows, we set $L := H^2(X, \mathbf{Z})$, $L^G := \{x \in L | g^*x = x \text{ for all } g \in G\}$, and $L_G := (L^G)^\perp$ in L. This L is the unique even unimodular lattice of index $(3, 19)$. We also denote by S_X the Néron-Severi lattice and by T_X $(:= S_X^\perp$ in $L)$, the transcendental lattice. Since G is simple and non-commutative, we have $G = [G, G]$. In particular, G acts on X symplectically. Therefore L^G contains both T_X and the invariant ample classes under G, namely the pull back of ample classes of X/G. In addition, since G is maximum among finite symplectic group actions [Mu1], G is normal in $\tilde{G}$ and the quotient group $\tilde{G}/G$ acts faithfully on $H^{2,0}(X) = \mathbf{C}\omega_X$. In particular, $\tilde{G}/G$ is a cyclic group of order I such that the Euler function $\varphi(I)$ divides rank T_X [Ni].

Claim (2.1).

(1) $\operatorname{rank} L^G = 3$. *In particular,* $\operatorname{rank} T_X = 2$ *and (up to scalar) there is exactly one* G*-invariant algebraic cycle class* H. *Moreover, this class* H *is ample and is also invariant under* $\tilde{G}$.

(2) $|\tilde{G}/G|$ *is either* 1, 2, 3, 4 *or* 6.

Proof. The equality $\operatorname{rank} L^G = 3$ is a special case of a general formula of Mukai. Here for the convenience of readers, we shall give a direct argument along [Mu1, Proposition 3.4]. Let us consider the natural representaion ρ of G on the cohomology ring of X

$$\tilde{L} := \oplus_{i=0}^{4} H^i(X, \mathbf{Z}) = H^0(X, \mathbf{Z}) \oplus L \oplus H^4(X, \mathbf{Z}).$$

Then, by the representation theory of finite groups and by the Lefschetz $(1,1)$-Theorem, one has

$$2 + \operatorname{rank} L^G = \operatorname{rank} \tilde{L}^G = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(\rho(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{top}}(X^g).$$

Here, the terms in the last sum are calculated by Nikulin [Ni] as follows:

$$\chi_{\mathrm{top}}(X^g) = 24, 8, 6, 4, 4, 2, 3, 2$$

if

$$\operatorname{ord}(g) = 1, 2, 3, 4, 5, 6, 7, 8.$$

Observe also that $n_1 = 1$, $n_2 = 21$, $n_3 = 56$, $n_4 = 42$, $n_7 = 48$ and $n_j = 0$ for other j if $G = L_2(7)$, where n_d denotes the cardinality of the elements of order d in G. Now combining all of these together, we obtain

$$2 + \operatorname{rank} L^G = \frac{1}{168}(24 + 8 \times 21 + 6 \times 56 + 4 \times 42 + 3 \times 48) = 5.$$

The remaining assertions now follow from the facts summarized before Claim (2.1). □

Remark (2.2). By (2.1)(1), K3 surfaces with $L_2(7)$-action are of the maximum Picard number 20. By a similar case-by-case calculation, one can also show that the invariant lattices are positive definite and of rank 3 if a K3 surface admits one of the eleven maximum symplectic group actions listed in [Mu1]. In particular, one has that

(1) the invariant lattices (tensorized by $\mathbf{R}$) have the hyperkähler three-space structure;

(2) such *algebraic* K3 surfaces are of the maximum Picard number 20 and are then at most countably many by [SI].

It would be very interesting to describe all of such (algebraic) K3 surfaces as rational points of the twister spaces corresponding to the invariant lattices (tensorized by $\mathbf{R}$). □

Next we determine the discriminant of L^G.

Key Lemma. $|\det L^G| = 196$.

The proof of Key Lemma will be given after Claim (2.6). Technically, this is the most crucial step and the next embedding Theorem due to Kondo is the most important ingredient in our proof of Key Lemma:

Theorem (2.3) [Ko1]. *Under the notation explained in Section 1, one has the following:*

(1) *For a given finite symplectic action H on a $K3$ surface, there exists a non-Leech Niemeier lattice N such that $L_H \subset N$. Moreover, the action of H extends to an action on N so that $L_H \simeq N_H$ and that N^H contains a simple root.*
(2) *This group action of H on N preserves the distinguished Weyl chamber $\mathcal{C}$ and the natural homomorphism $H \to S(N)$ is injective.* $\square$

Corollary (2.4). *Under the notation of Theorem (2.3), one has:*

(1) $\operatorname{rank} N^H = \operatorname{rank} L^H + 2$. *In particular,* $\operatorname{rank} N^G = \operatorname{rank} L^G + 2 = 5$.
(2) $|\det N^H| = |\det L^H|$. $\square$

Proof. Since $\operatorname{rank} N^H = 24 - \operatorname{rank} N_H$ and $\operatorname{rank} L^H = 22 - \operatorname{rank} L_H$, the first part of the assertion (1) follows from $N_H \simeq L_H$. Now the last part of (1) follows from (2.1). Recall that L and N are unimodular and the embeddings $L^H \subset L$ and $N^H \subset N$ are primitive. Then $|\det L^H| = |\det L_H|$ and $|\det N^H| = |\det N_H|$. Combining these with $N_H \simeq L_H$, one obtains $|\det N^H| = |\det L^H|$. $\square$

Let us return back to our original situation and determine the Niemeier lattice N for our G. Note that N is not the Leech lattice by (2.3)(1).

Claim (2.5). *The Niemeier lattice N in Theorem (2.3) for $G = L_2(7)$ is $N(A_1^{\oplus 24})$.*

Proof. By Theorem (2.3)(2), $168 = |G|$ divides $|S(N)|$. Thus, N is either $N(A_1^{\oplus 24})$ or $N(A_3^{\oplus 8})$ by (1.3). Suppose that the latter case occurs. Then, by (1.3), we have $G \subset C_2 \rtimes (C_2^{\oplus 3} \rtimes L_3(2))$. Since G is simple, a normal subgroup $G \cap C_2$ is trivial, i.e. $G \subset C_2^{\oplus 3} \rtimes L_3(2)$. Again, for the same reason, one has $G \subset L_3(2)$ (and in fact equal). The Dynkin diagram $A_3^{\oplus 7}$, the complement of the identity component in $A_3^{\oplus 8}$, consists of the 21 simple roots r_{i1}, r_{i2}, r_{i3} $(1 \le i \le 7)$ such that $(r_{i1}, r_{i2}) = (r_{i2}, r_{i3}) = 1$ but $(r_{i1}, r_{i3}) = 0$, $(r_{ik}, r_{jl}) = 0$ if $i \ne j$. Therefore the action G on these 21 simple roots satisfies $g(r_{i2}) = r_{g(i)2}$, where g in the right hand side is regarded as an element of the permutation of the seven components. Therefore $g(r_{i1})$ is either $r_{g(i)1}$ or $r_{g(i)3}$. Thus, G is embedded into a subgroup of the permutation subgroup $C_2^{\oplus 7} \rtimes S_7$ of the 14 simple roots r_{i1}, r_{i3}. Here, the indices $1, 3$ are so labelled that $\sigma \in S_7$ acts as $\sigma(r_{i1}) = r_{\sigma(i)1}$ and $\sigma(r_{i3}) = r_{\sigma(i)3}$ and the i-th factor of $C_2^{\oplus 7}$ acts as a permutation of r_{i1} and r_{i3}. Since G is simple and can not be embedded in $C_2^{\oplus 7}$, we have $G \subset S_7$. Therefore, $g(r_{i1}) = r_{g(i)1}$ and $g(r_{i3}) = r_{g(i)3}$. In conclusion, the orbits of the action G on the 24 simple roots are $\{r_{01}\}$, $\{r_{02}\}$, $\{r_{03}\}$, $\{r_{i1} | 1 \le i \le 7\}$, $\{r_{i2} | 1 \le i \le 7\}$, $\{r_{i3} | 1 \le i \le 7\}$. In particular, the 24 simple roots are divided into exactly 6 G-orbits. Since these 24 roots generate the

Niemeier lattice $N = N(A_3^{\oplus 8})$ over $\mathbf{Q}$, we have then rank $N^G = 6$, a contradiction to (2.4)(1). Hence the Niemeier lattice for our G is $N(A_1^{\oplus 24})$. $\square$

From now we set $N := N(A_1^{\oplus 24})$. By (2.5) and (1.4), we have

$$G \subset M_{24} \subset S_{24} = \mathrm{Aut}_{\mathrm{set}}(R).$$

Here $R := \{r_i\}_{i=1}^{24}$ is the set of the simple roots of N and the last inclusion is the natural one explained in (1.4). This allows us to use the table of the cyclic types of elements of M_{24} given in [EDM] for its action on R. One may also talk about the orbit decomposition type of the action of G on R. Although we donot know much about how G is embedded in M_{24}, we can say at least the following:

Claim (2.6) (cf. [Mu2]). *The orbit decomposition type of R by G is either*

$$[14, 1, 1, 7, 1] \textit{ or } [8, 7, 1, 7, 1].$$

Proof. Since $N^G = 5$ by (2.4)(1), the 24 simple roots of N are divided into exactly 5 G-orbits. (See the last argument of the Claim (2.5).) Set the orbit decomposition type as $[a, b, c, d, e]$. Then $a+b+c+d+e = 24$ and each entry is less than 21. In addition, since G is simple and contains an element of order 7, if $a \leq 6$ then $a = 1$, for otherwise the natural non-trivial representation $G \to S_a$ would have a non-trivial kernel. Moreover, if $a \geq 7$, then a divides $168 = |G|$. This is because the action of G on each orbit is, by the definition, transitive. Therefore a is either 1, 7, 8, 12 or 14. If $a = 12$, then an element of order 7 in G has already 5 fixed points in this orbit. However, by [EDM], the cycle type of order 7 element in M_{24} is $(7)^3(1)^3$ and therefore has only 3 fixed points, a contradiction. Hence a is either 1, 7, 8 or 14. Clearly the same holds for b, c, d, e. Now by combining these, together with the equality $a + b + c + d + e = 24$, we obtain the result. $\square$

Proof of Key Lemma. By (2.4)(2), we may calculate $|\mathrm{det} N^G|$ instead. Let us renumber the 24 simple roots according to the orbit decompositions found in (2.6):

$$\{r_1, \cdots, r_{14}\} \cup \{r_{15}\} \cup \{r_{16}\} \cup \{r_{17}, \cdots, r_{23}\} \cup \{r_{24}\} - (*)$$

or

$$\{r_1, \cdots, r_8\} \cup \{r_9, \cdots, r_{15}\} \cup \{r_{16}\} \cup \{r_{17}, \cdots, r_{23}\} \cup \{r_{24}\} - (**).$$

Consider the case $(*)$ first. Recall that rank $N^G = 5$ and $N^G = N \cap (A_1^{\oplus 24})^G \otimes \mathbf{Q}$. Then $b_1 = \sum_{i=1}^{14} r_i$, $b_2 = r_{15}$, $b_3 = r_{16}$, $b_4 = \sum_{i=17}^{23} r_i$ and $b_5 = r_{24}$ form the basis of $(A_1^{\oplus 24})^G$. Moreover, by (1.4), we see that $N^G/(A_1^{\oplus 24})^G$ consists of the elements of the form $\sum_{i \in I} b_i/2$, where the set of the simple roots $\{r_j\}$ appearing in the sum $\sum_{i \in I} b_i/2$ is either R, $\emptyset$, an Octad, complement of an Octad, or a Dodecad. However, by the shape of the orbit decomposition, there are no cases where a Dodecad appears. Therefore, in order to get an integral basis of N^G we may find out all the Octads and their complements appearing in the forms above.

Claim (2.7). *By reordering the three 1-element orbits if necessary, the union of the fourth and fifth orbits* $\{r_{17}, r_{18}, \cdots, r_{23}, r_{24}\}$ *forms an Octad.*

Proof. Let $\alpha \in G$ be an element of order 7. Then the cycle type of α (on R) is $(7)^3(1)^3$ [EDM]. In particular, a simple root x forms a 1-element orbit if $\alpha^k(x) = x$ for some k with $1 \le k \le 6$. Moreover, one can adjust the numbering of the roots in the fourth orbit $\{r_{17}, r_{18}, \cdots, r_{23}\}$ so as to be that $\alpha(r_i) = r_{i+1}$ $(17 \le i \le 22)$ and $\alpha(r_{23}) = r_{17}$. Let us consider the 5-element set $S := \{r_{17}, r_{18}, \cdots, r_{21}\}$. Then by (1.5), there is an Octad O such that $S \subset O$. We shall show that $O = \{r_{17}, r_{18}, \cdots, r_{23}, r_{24}\}$. For this purpose, assuming first that $r_{22}, r_{23} \notin O$, we shall derive a contradiction. Under this assumption, one has $\{r_{19}, r_{20}, r_{21}\} \subset O \cap \alpha^2(O)$ and $(O \cap \alpha^2(O)) \cap \{r_{17}, r_{18}, r_{22}, r_{23}\} = \emptyset$. (Here for the last equality we used the fact that $\alpha^2(r_{22}) = r_{17}$ and $\alpha^2(r_{23}) = r_{18}$.) The last equality also implies that $O \ne \alpha^2(O)$. Let us consider the symmetric difference $D = (O - \alpha^2(O)) \cup (\alpha^2(O) - O)$. Then $|D|$ must be either 8, 12, 16 whence $|O \cap \alpha^2(O)| = 4$ (See the proof of (1.5)). Thus, there exists an $x \in R$ such that $O \cap \alpha^2(O) = \{r_{19}, r_{20}, r_{21}, x\}$ and one has

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, x, y, z\}.$$

Here none of x, y, z lies in the fourth orbit. Since $x \in \alpha^2(O)$, one has either $x = \alpha^2(y)$ or $x = \alpha^2(x)$ (by changing the role of y and z if neccesary). In each case, we have $\alpha^2(z) \ne z$, whence $\alpha^k(z) \ne z$ for all k with $1 \le k \le 6$. In particular, z is in the first orbit.

Consider first the case where $x = \alpha^2(y)$. In this case, both x and y belong to the first orbit. Let us rename the elements in the first orbit so as to be that $\alpha(r_i) = r_{i+1}$ $(1 \le i \le 6)$, $\alpha(r_7) = r_1$; $\alpha(r_{7+i}) = r_{i+8}$ $(1 \le i \le 6)$, $\alpha(r_{14}) = r_8$ and that $y = r_1$ and $x = r_3$. Then, we have

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_3, r_1, z\},$$

and

$$\alpha^3(O) = \{r_{20}, r_{21}, r_{22}, r_{23}, r_{17}, r_6, r_4, \alpha^3(z)\}.$$

Considering the symmetric difference of O and $\alpha^3(O)$ as before, one finds that $O \cap \alpha^3(O)$ is a 4-element set. Therefore $|\{r_1, r_3, z\} \cap \{r_4, r_6, \alpha^3(z)\}| = 1$. Combining this with $\alpha^3(z) \ne z$, one has either $r_1 = \alpha^3(z)$, $r_3 = \alpha^3(z)$, $z = r_4$, or $z = r_6$. Hence O satisfies one of the following four:

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_1, r_3, r_5\} - (1)$$

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_1, r_3, r_7\} - (2)$$

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_1, r_3, r_4\} - (3)$$

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_1, r_3, r_6\} - (4).$$

In the case (1), one calculates

$$\alpha^2(O) = \{r_{19}, r_{20}, r_{21}, r_{22}, r_{23}, r_3, r_5, r_7\}$$

and has then $O \cap \alpha^2(O) = \{r_{19}, r_{20}, r_{21}, r_3, r_5\}$. In particular, the two Octads O and $\alpha^2(O)$ share 5 elements in common. Then, by the Steiner property (1.5), we would have $O = \alpha^2(O)$, a contradiction. By considering $O \cap \alpha(O)$ in the cases (2), (3) and $O \cap \alpha^2(O)$ in the case (4), we can derive a contradiction in the same manner, too. Thus, the case $x = \alpha^2(y)$ is impposible.

Next we consider the case where $\alpha^2(x) = x$. In this case, this x forms a 1-element orbit and satisfies $\{r_{18}, r_{19}, r_{20}, r_{21}, x\} \subset O \cap \alpha(O)$. However, the Steiner property would then imply $O = \alpha(O)$, whence $O = \alpha^2(O)$, a contradiction.

Therefore, the Octad O satisfies either $r_{22} \in O$ or $r_{23} \in O$, i.e.

$$\{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}\} \subset O,$$

or

$$\{r_{23}, r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\} \subset O.$$

Then one has either

$$\{r_{18}, r_{19}, r_{20}, r_{21}, r_{22}\} \subset O \cap \alpha(O),$$

or

$$\{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}\} \subset O \cap \alpha(O).$$

Hence, by the Steiner property, we have $O = \alpha(O)$, whence $O = \alpha^k(O)$ for all k. This implies that the Octad O is of the form

$$O = \{r_{17}, r_{18}, r_{19}, r_{20}, r_{21}, r_{22}, r_{23}, x\}$$

for some root x. Since $O = \alpha(O)$, we have also $\alpha(x) = x$. Hence this x forms a 1-element orbit set. $\square$

By this Claim, one has $b_6 := (b_4 + b_5)/2 \in N^G$ and also $b_7 := (b_1 + b_2 + b_3)/2 \in N^G$. By the remark before Claim (2.7) and the Steiner property (1.5), we also see that there are no other Octads appearing in the sum $\sum_{i \in I} b_i/2$. Since $\sum_{i=1}^{5} b_i/2 = b_6 + b_7$, the seven elements $b_1, \cdots, b_7$ then generate N^G over $\mathbf{Z}$. Moreover, since $b_1 = 2b_7 - b_2 - b_3$ and $b_4 = 2b_6 - b_5$, we finally see that b_7, b_2, b_3, b_6, b_5 form an integral basis of N^G. Using $(r_i, r_j) = -2\delta_{ij}$, we find that the intersection matrix of N^G under this basis is given as A below:

$$A = \begin{pmatrix} -8 & -1 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & -1 & -2 \end{pmatrix}.$$

Let us consider next the case $(**)$. Since G acts on the first 8-element orbit transitively, for each root r_i in the first orbit, one can find an element $\alpha' \in G$ of order 7 such that $\alpha'(r_i) \neq r_i$. Now, by the same argument based on the fact that the cycle type

of order 7 element is $(7)^3(1)^3$ and the Steiner property (1.5) (together with the remark above), one finds that (after reordering the two 1-element orbits) the union of the second and third orbits and the union of the fourth and fifth orbits are both Octads. This also implies that the first orbit is an Octad. Then again by the same argument as in the previous case, one can easily see that the five elements $b_1 := \sum_{i=1}^{8} r_i/2$, $b_2 := \sum_{i=9}^{16} r_i/2$, $b_3 = r_{16}$, $b_4 = \sum_{i=17}^{24} r_i/2$, and $b_5 = r_{24}$ form an integral basis of N^G in the second case, and that the intersection matrix of N^G under this basis is given as B above. Clearly $|\mathrm{det} N^G| = 196$ in both cases. This proves Key Lemma. $\square$

Next we shall study possible extensions $G \subset \tilde{G}$. Recall that we have already shown that $\tilde{G}/G \simeq \mu_I$, where I is either 1, 2, 3, 4 or 6, and that $\tilde{G}/G$ acts faithfully on T_X. Set $\tilde{G}/G = \langle \tau \rangle$.

The next Lemma is valid for any $G \subset \tilde{G}$ if $\tilde{G}/G$ acts faithfully on T_X and if rank $T_X = 2$.

Lemma (2.8).

(1) *Assume that* $\mathrm{ord}(\tau) = 3$. *Then, as* $\mathbf{Z}[\tau^*]$*-modules, one has* $T_X \simeq \mathbf{Z}[x]/(x^2+x+1)$, *where* τ^* *acts on the right hand side by the multiplication by* x. *In particular, one can take an integral basis* e_1, e_2 *of* T_X *such that* $\tau^*(e_1) = e_2$ *and* $\tau^*(e_2) = -(e_1 + e_2)$. *Moreover, under this basis, the intersection matrix of* T *is of the form* $((e_i, e_j)) = \begin{pmatrix} 2m & -m \\ -m & 2m \end{pmatrix}$.

(2) *Assume that* $\mathrm{ord}(\tau) = 4$. *Then, as* $\mathbf{Z}[\tau^*]$*-modules, one has* $T_X \simeq \mathbf{Z}[x]/(x^2 + 1)$, *where* τ^* *again acts on the right hand side by the multiplication by* x. *In particular, one can take an integral basis* e_1, e_2 *of* T_X *such that* $\tau^*(e_1) = e_2$ *and* $\tau^*(e_2) = -e_1$. *Moreover, under this basis, the intersection matrix of* T *is of the form* $((e_i, e_j)) = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix}$.

Proof. The first part of the two assertions is due to the fact that $\mathbf{Z}[\zeta_3]$ and $\mathbf{Z}[\zeta_4]$ are both PID. (For more detail, see for example [MO].) By taking an integral basis of T_X corresponding to 1 and x (in the right hand side), one obtains the desired representation of the action of τ^*. Now combining this with $(\tau^*(a), \tau^*(b)) = (a, b)$, we get the intersection matrix as claimed. $\square$

The next Claim completes the first assertion of the main Theorem:

Claim (2.9). $I \neq 6$.

A similar method is exploited in [Ko2] and [OZ] in other settings with somewhat different flavours and will be also adopted in the next Claim (2.10).

Proof. Assuming to the contrary that $\tilde{G}/G = \langle g \rangle \simeq \mu_6$, we shall derive a contradiction. By (2.1), one has $L^G \supset T_X \oplus \mathbf{Z}H$, where H is the primitive ample class invariant under G. Set $(H^2) = 2n$. Since T_X is primitive in L^G, we can choose an integral basis of L^G

as e_1, e_2 and $e_3 = (aH + be_1 + ce_2)/\ell$, where e_1 and e_2 are the integral basis of T_X found in (2.8)(1) applied for $\tau := g^2$ and ℓ and a, b, c are integers such that $(\ell, a) = 1$. Then

$$L^G/(T_X \oplus \mathbf{Z}H) = \langle \overline{e_3} \rangle \simeq C_\ell,$$

where $\overline{e_3} = e_3 \bmod (T_X \oplus \mathbf{Z}H)$. Since H is also stable under $\tilde{G}$, we have $\tau^*(\overline{e_3}) = \overline{e_3}$ and

$$\tau^*(be_1 + ce_2)/\ell \equiv (be_1 + ce_2)/\ell \bmod T_X.$$

On the other hand, by the choice of e_1, e_2, we calculate

$$\tau^*(be_1 + ce_2)/\ell = (-ce_1 + (b - c)e_2)/\ell.$$

Therefore, $b \equiv -c$ and $c \equiv b - c \bmod \ell$. In particular, $b \equiv -c$ and $3b \equiv 3c \equiv 0 \bmod \ell$. This, together with the primitivity of $\mathbf{Z}H$ in L^G, implies that $\ell = 1$ or 3, that is, $[L^G : T_X \oplus \mathbf{Z}H] = 1$ or 3.

If $\ell = 1$, we have $L^G = T_X \oplus \mathbf{Z}H$ and $196 = 6m^2n$. However 6 is not a divisor of 196, a contradiction.

Consider the case $\ell = 3$. Then, (by using the primitivity of H in L^G and by adding an element of T_X to e_3 if necessary), we can take one of $(\pm H \pm (e_2 - e_3))/3$ as e_3. Put $\sigma := g^3$. Then $\sigma^* H = H$ and $\sigma^*|T_X = -id$. Using these two equalities, we calculate

$$\sigma^*(e_3) = \sigma^*((\pm H \pm (e_2 - e_3))/3) = (\pm H \mp (e_2 - e_3))/3.$$

However, one would then have

$$\pm 2(e_2 - e_3)/3 = e_3 - \sigma^*(e_3) \in L^G,$$

a contradiction to the primitivity of T_X in L^G. Hence $I \neq 6$. $\square$

From now, we consider the maximum case $\tilde{G}/G = \langle \tau \rangle \simeq \mu_4$.

Claim (2.10). $(H^2) = 4$.

Proof. As in (2.9), one has $L^G \supset T_X \oplus \mathbf{Z}H$, where$H$ is the primitive ample class invariant under G. Set $(H^2) = 2n$. Since T_X is primitive in L^G, we can choose an integral basis of L^G as e_1, e_2 and $e_3 = (aH + be_1 + ce_2)/\ell$, where e_1 and e_2 are the integral basis of T_X found in (2.8)(2) and ℓ and a, b, c are integers such that $(\ell, a) = 1$. Then, as in (2.9), we have

$$L^G/(T_X \oplus \mathbf{Z}H) = \langle \overline{e_3} \rangle \simeq C_\ell,$$

where $\overline{e_3} = e_3 \bmod (T_X \oplus \mathbf{Z}H)$. Since H is also stable under $\tilde{G}$, we have $\tau^*(\overline{e_3}) = \overline{e_3}$ and

$$\tau^*(be_1 + ce_2)/\ell \equiv (be_1 + ce_2)/\ell \bmod T_X.$$

On the other hand, by the choice of e_1, e_2, we calculate

$$\tau^*(be_1 + ce_2)/\ell = (be_2 - ce_1)/\ell.$$

Therefore, $b \equiv c$ and $c \equiv -b \bmod \ell$. In particular, $b \equiv c$ and $2b \equiv 2c \equiv 0 \bmod \ell$. This, together with the primitivity of $\mathbf{Z}H$ in L^G, implies that $\ell = 1$ or 2, that is, $[L^G : T_X \oplus \mathbf{Z}H] = 1$ or 2.

In the first case, we have $L^G = T_X \oplus \mathbf{Z}H$ and $196 = 8m^2n$. However 8 is not a divisor of 196, a contradiction.

In the second case, we have $2^2 \cdot 196 = 8m^2n$, i.e. $m^2n = 2 \cdot 7^2$. Then (m, n) is either $(1, 2 \cdot 7^2)$ or $(7, 2)$. In the first case we have $X = X_4$ by the result of Shioda and Inose [SI], where X_4 is the minimal resolution of $(E_{\sqrt{-1}} \times E_{\sqrt{-1}})/\langle \mathrm{diag}(\sqrt{-1}, -\sqrt{-1})\rangle$. However, according to the explicit description of $\mathrm{Aut}(X_4)$ by Vinberg [Vi], X_4 has no automorphism of order 7, a contradiction. Therefore, only the second case can happen and one has $(H^2) = 2n = 4$ (and $T_X = \mathrm{diag}(14, 14)$). $\square$

Now the following Claim will complete the proof of the main Theorem.

Claim (2.11). *$(X, \tilde{G})$ is isomorphic to $(X_{168}, L_2(7) \times \mu_4)$ defined in the Introduction.*

Proof. Since $S_X^G = \mathbf{Z}H$, $|H|$ has no fixed components. Indeed, the fixed part of $|H|$ must be also G-stable but is of negative definite [SD]. Therefore, the ample linear system $|H|$ is free [ibid.]. Note that $\dim|H| = 3$ by the Riemann-Roch formula and the fact $(H^2) = 4$. Then $|H|$ defines a morphism $\Phi := \Phi_{|H|} : X \to \mathbf{P}^3$. This Φ is either an embedding to a quartic surface S or a finite double cover of an integral quadratic surface Q. Note that $\tilde{G}$ acts on the image as a projectively linear transformation. Moreover, the action of G on the image is faithful even in the second case, because G is simple. Recall that the degrees of the projectively linear irreducible representations of G are $1, 3, 4, 6, 7, 8$ [ATLAS] and that the two 3-dimensional irreducible representations are transformed by the outer automorphism of G. Then the action of G on the image is induced by the irreducible decomposition $\mathbf{C}^4 = V_1 \oplus V_3$ or $\mathbf{C}^4 = V_4$. (Note that the action of G on $\mathbf{P}^3$ is linearlized in the first case but not in the second case. More precisely, in the second case, only the action of $\mathrm{SL}(2, \mathbf{F}_7)$, i.e. the central extension of G corresponding to the Schur multiplier 2, is linearlized.)

Let us first consider the second case. By [Ed, Pages 198 - 200 and Page 166], G has no invariant hypersurface of degree 2 but only one invariant hypersurface of degree 4:

$$f := x_0^4 + 6\sqrt{2}x_0x_1x_2x_3 + x_1x_2^3 + x_2x_3^3 + x_3x_1^3 = 0,$$

where the homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ are chosen in such a way that an order 7-element of G, say α, is represented by the following diagonal matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_7^6 & 0 & 0 \\ 0 & 0 & \zeta_7^3 & 0 \\ 0 & 0 & 0 & \zeta_7^5 \end{pmatrix}.$$

Thus, Φ is an embedding and one has $S = (f = 0)$. Recall that $\tilde{G}$ fits in with the exact sequence

$$1 \to G \to \tilde{G} \to \mu_4 \to 1,$$

where the last map is the representation of $\tilde{G}$ on $H^0(S, \Omega_S^2) = \mathbf{C}\omega_S$. Let $g \in \tilde{G}$ be a lift of $\zeta_4 \in \mu_4$. Then $g^*\omega_S = \zeta_4\omega_S$. Since G is a normal subgroup of $\tilde{G}$, one can define an element $c_g \in \mathrm{Aut}_{\mathrm{group}}(G)$ by $G \ni x \mapsto g^{-1}xg \in G$ for all $x \in G$. Since $\mathrm{Out}(G) \simeq C_2$ [ATLAS], $(c_g)^2$ is then an inner automorphism of G, i.e. there exists an element $y \in G$ such that $g^{-2}xg^2 = y^{-1}xy$ for all $x \in G$. Set $k = g^2y^{-1}$. Then one has $k^{-1}xk = x$ for all $x \in G$, $k^*\omega_S = -\omega_S$ and $2|\mathrm{ord}(k)$. Therefore, replacing k by k^{2l+1} if necessary, one obtains an element $h \in \tilde{G}$ such that $h^{-1}xh = x$ for all $x \in G$, $h^*\omega_S = -\omega_S$ and $\mathrm{ord}(h) = 2^n$. Choose a representative (h_{ij}) of h in $\mathrm{GL}(4, \mathbf{C})$. Then for A above one has $(h_{ij})A(h_{ij})^{-1} = cA$ ($c \in \mathbf{C}$) in $\mathrm{GL}(4, \mathbf{C})$. This implies $c^{2^n} = 1$ and $\{1, \zeta_7^6, \zeta_7^3, \zeta_7^5\} = \{c, c\zeta_7^6, c\zeta_7^3, c\zeta_7^5\}$. Thus $c = 1$ and one has $(h_{ij})A(h_{ij})^{-1} = A$, i.e. $(h_{ij})A = A(h_{ij})$ in $\mathrm{GL}(4, \mathbf{C})$. This readily implies that (h_{ij}) is also a diagonal matrix, and one may write that

$$(h_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & h_{11} & 0 & 0 \\ 0 & 0 & h_{22} & 0 \\ 0 & 0 & 0 & h_{33} \end{pmatrix}.$$

Then, by the shape of f and by the fact that $h^*f = c'f$ for some $c' \in \mathbf{C}$, one has $c' = 1$ and $h_{11}h_{22}h_{33} = 1$. However, this would yield $(h_{ij})^*f = f$ and $\det(h_{ij}) = 1$, thereby $h^*\omega_S = \omega_S$, a contradiction to the previous equality $h^*\omega_S = -\omega_S$. Hence the second case cannot happen.

Let us next consider the first case. Let us choose the homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ such that x_0 is the coordinate of V_1 and that x_i ($1 \le i \le 3$) are the coordinates of V_3 described as in the Introduction.

Let us first consider the case where Φ is a double covering. Write an equation of Q as

$$ax_0^2 + x_0f_1(x_1, x_2, x_3) + f_2(x_1, x_2, x_3) = 0.$$

Since G is simple and acts on x_0 as an identity, we have $g^*(f_1) = f_1$ and $g^*(f_2) = f_2$ for all $g \in G$. Since there are no non-trivial G-invariant linear and quadratic forms in three variables [Bu, Section 267], one has $f_1 = f_2 = 0$. However, then Q is not integral, a contradiction. Hence Φ is an embedding. Let us write an equation of S as

$$F = ax_0^4 + x_0^3f_1(x_1, x_2, x_3) + x_0^2f_2(x_1, x_2, x_3) + x_0f_3(x_1, x_2, x_3) + f_4(x_1, x_2, x_3) = 0.$$

Then $g^*(f_i) = f_i$ for all $g \in G$ and all $1 \le i \le 4$. Thus by [ibid.], we have $f_i = 0$ for all $1 \le i \le 3$, $f_4(x_1, x_2, x_3) = b(x_1x_2^3 + x_2x_3^3 + x_3x_1^3)$ and $F(x_0, x_1, x_2, x_3) = ax_0^4 + b(x_1x_2^3 + x_2x_3^3 + x_3x_1^3)$. Here $a \neq 0$ and $b \neq 0$, because S is non-singular. Therefore, by multiplying coordinates suitably, one may adjust the equation of S as

$$x_0^4 + x_1x_2^3 + x_2x_3^3 + x_3x_1^3 = 0.$$

Hence $S \simeq X_{168}$ and $L_2(7) \times \mu_4$ acts on S as described in the Introduction, where by the construction, the action $L_2(7)$ also coincides with the given action of G on X. Since S is a K3 surface and has no non-zero global holomorphic vector fields, the projectively linear automorphism group G'' of $S \subset \mathbf{P}^3$ is finite. This G'' satisfies $G'' \supset L_2(7) \times \mu_4$ and $G'' \supset \tilde{G}$. Thus $4 \cdot 168 || G''|$ and one has $|G''| = |L_2(7) \times \mu_4| = |\tilde{G}| = 4 \cdot 168$. Hence $\tilde{G} = G'' = L_2(7) \times \mu_4$ as projectively linear automorphism groups of S. Now we are done. $\square$

Remark (2.12).

(1) By the proof of (2.10), we have that $T_X = \mathrm{diag}(14, 14)$ if $|\tilde{G}/G| = 4$. In particular, $T_{X_{168}} = \mathrm{diag}(14, 14)$.

(2) Now one can easily check that the two K3 surfaces X_{168} and X'_{168} in the Introduction are not isomorphic to each other. Note that X'_{168} has a G-stable ample class H of degree 2. Therefore, if $T_{X'_{168}}$ is isomorphic to $T_{X_{168}} = \mathrm{diag}\ (14, 14)$, then $[L^G : T_{X'_{168}} \oplus \mathbf{Z}H]^2 = 2 \cdot 14^2/196 = 2$, a contradiction. $\square$

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Added in Proof

In this added in proof, we continue to employ the same notation, eg. $G = L_2(7)$. After submitting our note, we noticed the following Propositions. These answer questions asked by I. Dolgachev around January 2001:

Proposition 1. *In the main Theorem (1), one has* $|\tilde{G}/G| \neq 3$.

Proposition 2. *There are infinitely many non-isomorphic algebraic K3 surfaces* X *such that* $G \subset Aut(X)$.

In Proposition 2, we already know that there are at most countably many such algebraic K3 surfaces X (Remark (2.2)).

Proof of Proposition 1

Assuming to the contrary, we shall derive a contradiction. Recall that $\mathrm{rk} S_X = 20$ and $\mathrm{rk} T_X = 2$ if X is an algebraic K3 surface admitting an $L_2(7)$-action.

Claim 1. $\tilde{G} \simeq G \times \mu_3$.

Proof. Let $\tilde{g} \in \tilde{G}$ be an element such that $\tilde{g}^* \omega_X = \zeta_3 \omega_X$. Replacing $\tilde{g}$ by its power $\tilde{g}^n$ such that $(n, 3) = 1$, one may assume that $\mathrm{ord}(\tilde{g}) = 3^m$ for some positive integer m. Since G is a normal subgroup of $\tilde{G}$, one has $c_{\tilde{g}}(x) := \tilde{g}^{-1} x \tilde{g} \in G$ if $x \in G$, thereby $c_{\tilde{g}} \in \mathrm{Aut}(G)$. Since $\mathrm{Out}(G) = C_2$ and $\mathrm{ord}(\tilde{g}) = 3^m$, one has then $c_{\tilde{g}} \in \mathrm{Inn}(G)$, i.e. there is $y \in G$ such that $\tilde{g}^{-1} x \tilde{g} = y^{-1} x y$ for all $x \in G$. Now, replacing $\tilde{g}$ by $\tilde{g} y^{-1}$, one has $\tilde{g}^{-1} x \tilde{g} = x$ for all $x \in G$ and $\tilde{g}^* \omega_X = \zeta_3 \omega_X$. Then $\tilde{g}^3 \in G$ and is also in the center of G. Note that the center of G is $\{id.\}$ for G being simple, non-commutative. Then $\tilde{g}^3 = id.$ and $\tilde{g}$ gives a desired splitting of the exact sequence $1 \to G \to \tilde{G} \to \mu_3 \to 1$. $\square$

Claim 2. *T_X is isomorphic to* $\begin{pmatrix} 14 & -7 \\ -7 & 14 \end{pmatrix}$ *(and the degree of the primitive invariant polarization is* 12*).*

Proof. The argument is the same as in (2.10) but is based on the following three facts instead: Lemma (2.8)(1) (instead of (2)); $[L^G : T_X \oplus \mathbf{Z}H] = 3$ (In the course of proof of (2.9)); and the fact that X admits no automorphisms of order 7 if T_X is isomorphic to $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ [Vi]. Further details are left to the readers. $\square$

Let us next consider the irreducible decomposition of the natural linear action of G on $S_X \otimes \mathbf{C}$. We adopt the notation in [ATLAS]. We denote the irreducible representation of the character χ_i in [ibid.] by V_i. Then, $\dim V_i$ is 1, 3, 3, 6, 7, 8 if $i = 1, 2, 3, 4, 5, 6$.

Claim 3. *The irreducible decomposition of the natural action of G on $S_X \otimes \mathbf{C}$ is:*

$$S_X \otimes \mathbf{C} = V_1 \oplus V_4 \oplus V_4' \oplus V_5,$$

where V_4' is a copy of V_4.

Proof. Set $S_X \otimes \mathbf{C} = \oplus_{i=1}^{6} V_i^{\oplus n_i}$. Here we have $n_1 = 1$ by (2.1)(1) and also $n_2 = n_3$ as V_2 and V_3 are (complex) conjugate to each other. Then by counting the dimension, one has

$$20 = n_1 + 6n_2 + 6n_4 + 7n_5 + 8n_6.$$

Recall that for $g \in G$, one has

(1) $\chi_{\text{top}}(X^g) = 2 + \text{tr}(g^*|S_X) + \text{tr}(g^*|T_X) = 4 + \text{tr}(g^*|S_X)$

(2) $\chi_{\text{top}}(X^g)$ is 8, 6, 4, 3 if $\text{ord}(g) = 2, 3, 4, 7$.

The first equality is nothing but the Lefschetz fixed point formula and the second one is the result of Nikulin [Ni]. By applying these two formula and the character table in [ATLAS] for elements of G of order 2, 3, 4, 7 respectively, one obtains:

$$n_1 - 2n_2 + 2n_4 - n_5 = 4,$$

$$n_1 + n_5 - n_6 = 2,$$

$$n_1 + 2n_2 - n_5 = 0,$$

$$n_1 - n_2 - n_4 + n_6 = -1.$$

Combining all of these together, we find that $n_1 = 1$, $n_2 = n_3 = 0$, $n_4 = 2$, $n_5 = 1$, $n_6 = 0$. This gives the result. $\square$

Let τ be a generater of μ_3. We regard $\tau \in \tilde{G}$ through the isomorphism found in Claim 1. Since $\tau^{-1} g \tau = g$ for all $g \in G$ and τ is of order 3, one has $\tau(V_i) = V_i$ for each i and $\tau(V_4') = V_4'$. Then by Schur's Lemma, $\tau|V_i$, $\tau|V_i'$ are all scalar multiplications. Note that $\tau|V_1 = id.$ for $S_X^{\tilde{G}} \neq \{0\}$. Set $\tau|V_4 = \zeta_3^a$, $\tau|V_4' = \zeta_3^b$, and $\tau|V_5 = \zeta_3^c$, where $a, b, c \in \mathbf{Z}/3$. Since $\tau|S_X \otimes \mathbf{C}$ is defined over S_X, the multiplicities of eigenvalues ζ_3 and

ζ_3^2 of $\tau|S_X \otimes \mathbf{C}$ are the same. Therefore, $c = 0$ and $a+b = 0$, i.e. (a, b, c) is either $(0, 0, 0)$ or $(1, 2, 0)$.

Consider first the case $(a, b, c) = (0, 0, 0)$. Then $\tau^*|S_X = id.$ (and $\tau^*\omega_X = \zeta_3\omega_X$). However T_X would then be a 3-elementary lattice by [OZ2, Lemma (1.3)], a contradiction to Claim 2.

Let us consider next the case $(a, b, c) = (1, 2, 0)$. Let g be an order 2 element of G. Set $h := \tau g \in \tilde{G}$. Then h is of order 6 and satisfies $h^3 = g$. In particular, one has $X^h \subset X^g$. Here X^g is an 8-point set by [Ni]. Thus, X^h also consists of finitely many points (possibly empty), thereby $\chi_{\mathrm{top}}(X^h) \geq 0$. On the other hand, by using the Lefschetz fixed point formula, the fact $(a, b, c) = (1, 2, 0)$, Claim 3, and the character table [ATLAS], one calculates

$$\chi_{\mathrm{top}}(X^h) = 2 + \mathrm{tr}(h^*|S_X) + \mathrm{tr}(h^*|T_X)$$

$$= 2 + \{1 + \mathrm{tr}(g|V_4)(\zeta_3 + \zeta_3^2) + \mathrm{tr}(g|V_5) \cdot 1\} + (\zeta_3 + \zeta_3^2)$$

$$= 2 + 1 + 2 \cdot (-1) + (-1) \cdot 1 + (-1) = -1 < 0,$$

a contradiction to the previous inequality $\chi_{\mathrm{top}}(X^h) \geq 0$. Now we are done. $\square$

Proof of Proposition 2

Let Λ be the K3 lattice, i.e. the lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Choosing a marking $\tau : H^2(X_{168}, \mathbf{Z}) \simeq \Lambda$, we set $\Lambda_0 := \tau(H^2(X_{168}, \mathbf{Z})^G)$. This Λ_0 is a positive definite even lattice of rank 3 whose $\mathbf{R}$ linear extension is spanned by the image of the classes of the invariant ample class η, $\mathrm{Re}(\omega_{X_{168}})$ and $\mathrm{Im}(\overline{\omega}_{X_{168}})$. Fixing the Ricci flat Kähler metric g on X_{168} such that the cohomology class of the associated $(1,1)$-form is η and regarding $\Lambda_0 \otimes \mathbf{R}$ as a HK 3-space, one obtains the twister family $f : \mathcal{X} \to \mathbf{P}^1$ with $\mathcal{X}_0 = X_{168}$ (See for instance [Be, Exposé X]). This f is a smooth non-isotrivial family of (not necessarily algebraic) K3 surfaces $\mathcal{X}_t$. Denote by ω_t a nowhere vanishing holomorphic two form on $\mathcal{X}_t$ and by η_t the Kähler class on $\mathcal{X}_t$ associated with g. Let us fix a marking $\tilde{\tau} : R^2 f_* \mathbf{Z} \simeq \Lambda$ such that $\tilde{\tau}_0 = \tau$. (Here we used the fact that $\mathbf{P}^1$ is simply-connected.) By the construction, for each $t \in \mathbf{P}^1$, the HK 3-space $\Lambda_0 \otimes \mathbf{R}$ is spanned by the three vectors $\tilde{\tau}_t(\eta_t)$, $\tilde{\tau}_t(\omega_t)$ and $\tilde{\tau}_t(\overline{\omega}_t)$. In particular, we have $\rho(\mathcal{X}_t) \geq 19$ for all $t \in \mathbf{P}^1$. There are then infinitely many t such that $\rho(\mathcal{X}_t) = 20$ by [Og2]. Such $\mathcal{X}_t$ is necessarily algebraic by [SI] and $\tilde{\tau}_t(T_{\mathcal{X}_t})$ is a primitive sublattice of rank 2 of Λ_0. Using the marking $\tilde{\tau}$, let us define the (real) period map:

$$\iota \circ p : \mathbf{P}^1 \to \{[\omega] \in \mathbf{P}(\Lambda \otimes \mathbf{C}) | (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}$$

$$\simeq \{T \in \mathrm{Gr}^+(2, \Lambda \otimes \mathbf{R}) | T \,\text{is positive definite}\}.$$

Since p is a complex analytic map and $\mathbf{P}^1$ is compact, $\iota \circ p$ is finite. Therefore, for each rank two sublattice T (of Λ_0), there are at most finitely many $t \in \mathbf{P}^1$ such that $\tilde{\tau}_t(T_{\mathcal{X}_t}) = T$. Hence, by the global Torelli Theorem for K3 surfaces with the maximum Picard number 20 ([SI]), the family f contains infinitely many non-isomorphic algebraic K3 surfaces. Now the following Claim completes the proof of Proposition 2:

Claim. *$\mathcal{X}_t$ satisfies $G \subset Aut(\mathcal{X}_t)$ for all $t \in \mathbf{P}^1$.*

This Claim also shows that there are uncountably many (non-algebraic) K3 surfaces admitting $L_2(7)$-actions.

Proof. Since $G|(\tilde{\tau}_0)^{-1}(\Lambda_0) = \{id.\}$ and $\eta_t, \mathrm{Re}(\omega_t), \mathrm{Im}(\omega_t) \in (\tilde{\tau}_t)^{-1}(\Lambda_0 \otimes \mathbf{R})$, we see that $(\tilde{\tau}_t)^{-1} \circ (\tilde{\tau}_0) \circ G \circ (\tilde{\tau}_0)^{-1} \circ (\tilde{\tau}_t)$ is an effective Hodge isometry of $H^2(\mathcal{X}_t, \mathbf{Z})$. This action is also faithful, beacuse G is simple and $G|(\tilde{\tau}_0)^{-1}(\Lambda) \neq \{id.\}$. Hence $G \subset \mathrm{Aut}(\mathcal{X}_t)$ for each $t \in \mathbf{P}^1$ by the global Torelli Theorem for K3 surfaces. $\square$

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